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Covariant approach to equilibration in effective field theories

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Abstract

The equilibration of two coupled reservoirs is studied using a Green function approach which is suitable for future development with the closed time path method. The problem is solved in two parameterizations, in order to demonstrate the non-trivial issues of parameterization in both the intermediate steps and the interpretation of physical quantities. We use a covariant approach to find self-consistent solutions for the statistical distributions as functions of time. We show that by formally introducing covariant connections, one can rescale a slowly varying non-equilibrium theory so that it appears to be an equilibrium one, for the purposes of calculation. We emphasize the importance of properly tracking variable redefinitions in order to correctly interpret physical quantities.

1 Introduction

Effective field theory is an increasingly important tool in the repertoire of theoretical physics. The problem of the equilibration of statistical systems has been studied in many contexts [1, 2, 3, 4, 5, 6], but has not been formulated in the language of effective quantum field theories, where re-parameterizations of basic quantities can obscure the physics of the process.

Suppose we consider two systems with different mechanical and thermodynamical properties and couple these together. What is the outcome? Intuition tells us that such a system must achieve a state of thermodynamical equilibrium. However, it is not clear how this equilibrium is achieved using the language of effective field theory. In terms of particles, it is straightforward to picture the exchange of energy

between heat baths: there is a physical flow of point-like objects carrying momentum, which are exchanged until, eventually, there is an equal spectrum of energies on both sides of the interface. In a field theory however, one deals with quanta or modes of the field, so there is no direct notion of a flow of hot into cold or vice versa, only the occupation of states in a Fock space.

The procedure of identifying a functional effective field theory involves many variable redefinitions and partial summations. Each re-parameterization distances the system from any intuition one might have about its behaviour. Our goal in this paper is to understand the process of equilibration in the language of field theory in the context of a specific model. We consider a pair of essentially free scalar fields with different masses and different initial temperatures which are coupled minimally at an initial time. For definiteness we assume that the systems are in direct contact, rather than interacting through a boundary layer, so that the two systems essentially occupy the same space. Although it is convenient to focus on a specific model, our aim is to extract general principles about non-equilibrium field theory. We employ the methods of general covariance during field redefinitions, in order to track the meaning of the quantities which arise in the formulation, and also to find the optimal parameterization for solving the equations. As is often the case with covariance, this leads to important physical insights.

The notation and conventions are those of reference [8, 9], with n spatial dimensions and one time dimension. To simplify notation we shall use Schwinger's shorthand for the measures:

$$\begin{aligned} \int \frac{d^{n+1}k}{(2\pi)^{n+1}} &\equiv (dk) \\ \int \frac{d^n\mathbf{k}}{(2\pi)^n} &\equiv (d\mathbf{k}) \\ dV_x = dtd^n\mathbf{x} &\equiv (dx) \end{aligned} \tag{1}$$

and so on.

A non-equilibrium system remembers its history; it is not translationally invariant in time. It is convenient to parameterize Green functions in terms of even and odd variables over any interval from x to x':

$$\begin{aligned} \tilde{x} &= (x - x') \\ \overline{x} &= \frac{1}{2}(x + x'). \end{aligned} \tag{2}$$

The odd variables $\tilde{x}$ characterize translational degrees of freedom, while the even variables $\overline{x}$ represent inhomogeneous impediments to translational freedom, or alternatively, characterize local curvature in the dynamical and kinematical parameters.

The formulation we adopt is based on [8], which is built around Schwinger's closed time path quantum action principle. We do not need to formulate the full closed time path problem here, but our approach is easily generalizable for future problems. We concentrate on understanding the basic unitary processes which act in the system, and employ the language of sources and Green functions to describe the development.

2 The model

Consider two systems which are brought into contact, so as to interact through a minimal coupling which is not manifestly Z_2 ($\phi \to -\phi$) invariant. Once coupled, the systems influence one another. Since the systems are statistical mixtures of field modes, this influence can be viewed at two levels: as the microscopic interaction of

individual field solutions between the two systems, or as a change in the statistical averages of the two systems as a result of the microscopic interaction. It is at this latter level that one expects to achieve equilibrium. The model we choose begins as a free field theory, composed of a thermal mixture of quanta. This initial state is clearly idealized since one normally requires some form of interaction in order to achieve thermal equilibrium in the first place. In the interests of simplicity, we suppress all such non-linear interactions and assume that the initial state is two effectively free thermal baths.

We formulate the physical system using the Schwinger action principle. The uncoupled systems have the action operators:

$$\begin{aligned} S_1 &= \int (dx) \left\{ (\partial_\mu \hat{\Phi}_1)^* (\partial^\mu \hat{\Phi}_1) + \hat{m}_1^2 \right\} \\ S_2 &= \int (dx) \left\{ (\partial_\mu \hat{\Phi}_2)^* (\partial^\mu \hat{\Phi}_2) + \hat{m}_2^2 \right\}. \end{aligned} \tag{3}$$

Before coupling the equations of motion give us the eigenvalue equation:

$$\hat{\mathcal{O}}\hat{\Phi} = 0 \quad \text{or} \quad \begin{pmatrix} -\Box + \hat{m}_1^2 & 0 \\ 0 & -\Box + \hat{m}_2^2 \end{pmatrix} \begin{pmatrix} \hat{\Phi}_1 \\ \hat{\Phi}_2 \end{pmatrix} = 0. \tag{4}$$

The Green function $\hat{\mathcal{G}}$ for the operator $\hat{\mathcal{O}}$ satisfies the equation $\hat{\mathcal{O}}\hat{\mathcal{G}} = I$. We write the Green function as the sum of a particular integral and a complementary function

$$\hat{\mathcal{G}} = \hat{G} + 2\pi i \hat{\mathcal{F}} \tag{5}$$

which satisfy

$$\hat{\mathcal{O}}\hat{G} = I\,; \quad \hat{\mathcal{O}}\hat{\mathcal{F}} = 0\,. \tag{6}$$

In most pure state cases one sets $\hat{\mathcal{F}} = 0$ and chooses a special solution for the Green function G which satisfies certain specific boundary conditions (retarded, advanced, etc.). The complementary function $\hat{\mathcal{F}}$ becomes important in statistical many particle systems where mixtures of fields are significant. However, it is not possible to associate specific boundary conditions with the free particle solutions that are contained in the complementary piece (free particle solutions do not respect time-specific boundary conditions, since plane-wave solutions have no beginning or end). The coefficients of plane wave solutions must therefore change in time in order to follow the development of the system, in a non-equilibrium many particle system. This time dependence is discussed in detail in the next sections. We obtain,

$$\hat{G} = \begin{pmatrix} \frac{1}{-\Box + \hat{m}_1^2} & 0 \\ 0 & \frac{1}{-\Box + \hat{m}_2^2} \end{pmatrix} \tag{7}$$

and

$$\hat{\mathcal{F}} = \begin{pmatrix} \hat{f}_1(k_0)\delta(\hat{\chi}_1) & 0 \\ 0 & \hat{f}_2(k_0)\delta(\hat{\chi}_2) \end{pmatrix}. \tag{8}$$

Although nothing formally precludes off diagonal terms in $\hat{\mathcal{F}}$ where the upper right contribution is proportional to $\delta(\hat{\chi}_1)$ and the lower left contribution is proportional to $\delta(\hat{\chi}_2)$, the boundary conditions tell us that there is no interaction between the two systems, so off-diagonal terms would not make sense.

Once coupled, the fields are influenced by their mutual interaction. Since this interaction leads to modifications to the constraints (mass shell) of the systems, we denote the coupled variables without carets:

$$\begin{aligned} S_1 &= \int (dx) \left\{ (\partial_\mu \Phi_1)^* (\partial^\mu \Phi_1) + \hat{m}_1^2 \Phi_1^* \Phi_1 + J_1^* \Phi_1 + J_1 \Phi_1^* \right\} \\ S_2 &= \int (dx) \left\{ (\partial_\mu \Phi_2)^* (\partial^\mu \Phi_2) + \hat{m}_2^2 \Phi_2^* \Phi_2 + J_2^* \Phi_2 + J_2 \Phi_2^* \right\} \end{aligned} \tag{9}$$

Both systems interact only through linear perturbations. In order to couple the systems together we choose:

$$\begin{aligned} J_1 &= m_p^2 \, \Phi_2 \\ J_2 &= m_p^2 \, \Phi_1, \end{aligned} \tag{10}$$

where m_p^2 is a squared polaron mass introduced for dimensional consistency so that each field is a source for the other. Starting from the total action,

$$S_{\rm tot} = (S_1 + S_2)\big|_{J_1 = m_p^2 \Phi_2, J_2 = m_p^2 \Phi_1}, \tag{11}$$

the operator equations of motion are,

$$\begin{aligned} (-\Box + \hat{m}_1^2)\Phi_1 &= -m_p^2 \Phi_2 \\ (-\Box + \hat{m}_2^2)\Phi_2 &= -m_p^2 \Phi_1. \end{aligned} \tag{12}$$

Note that the coupling destroys the diagonal structure of the equations of motion. We have:

$$\hat{\mathcal{O}} \Phi = J \Phi \tag{13}$$

With $\mathcal{O} = \hat{\mathcal{O}} - J$ we can write,

$$\mathcal{O}\Phi = 0 \quad \text{or} \quad \begin{pmatrix} -\Box + \hat{m}_1^2 & -m_p^2 \\ -m_p^2 & -\Box + \hat{m}_2^2 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = 0. \tag{14}$$

In order to find the corresponding Green function we diagonalize the matrix $\mathcal{O}$. The technique is standard. We solve the eigenvalue equation

$$\begin{pmatrix} -\Box + \hat{m}_1^2 & -m_p^2 \\ -m_p^2 & -\Box + \hat{m}_2^2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = (-\Box + m_\pm^2) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{15}$$

and compute the invertible unitary matrix of eigenvectors U. The diagonalized matrix is obtained from the similarity transformation,

$$\mathcal{O}' = (U^\dagger)^{-1} \mathcal{O} U^{-1} = U \mathcal{O} U^{-1} \tag{16}$$

The Green function matrix transforms according to the same rule as the operator itself:

$$\mathcal{G} \to \mathcal{G}' = U \, \mathcal{G} \, U^{-1}. \tag{17}$$

which preserves the relationship

$$\mathcal{O}\mathcal{G} = \mathcal{O}'\mathcal{G}' = I \tag{18}$$

The similarity transformation corresponds to a field transformation of the form,

$$\Phi_A = U_{AB} \, \varphi_B, \tag{19}$$

where $A, B = 1, 2$.

We proceed as follows. Solving the eigenvalue equation (15) for the eigenvalues gives,

$$\begin{aligned} m_\pm^2 &= \overline{m}^2 \pm \sqrt{\tilde{m}^4 + m_p^4} \\ &\equiv \overline{m}^2 \pm m_R^2 \,. \end{aligned} \tag{20}$$

with

$$\bar{m}^2 := \frac{1}{2}(\hat{m}_1^2 + \hat{m}_2^2)\,; \quad \tilde{m}^2 := \frac{1}{2}(\hat{m}_1^2 - \hat{m}_2^2) \tag{21}$$

Using the identities,

$$\begin{aligned} (m_\pm^2 - \hat{m}_1^2) &= -\tilde{m}^2 \pm m_R^2 \\ (m_\pm^2 - \hat{m}_2^2) &= \tilde{m}^2 \pm m_R^2 \,, \end{aligned} \tag{22}$$

the invertible unitary matrix of eigenvectors is easily calculated:

$$U = \frac{1}{\sqrt{m_p^4 + (\tilde{m}^2 - m_R^2)^2}} \begin{pmatrix} m_p^2 & (m_R^2 - \tilde{m}^2) \\ -(m_R^2 - \tilde{m}^2) & m_p^2 \end{pmatrix} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{23}$$

Using (14), (16) and (23) we obtain the diagonalized form of the operator $\mathcal{O}$:

$$\mathcal{O}' = \begin{pmatrix} -\Box + \hat{m}_+^2 & 0 \\ 0 & -\Box + \hat{m}_-^2 \end{pmatrix} . \tag{24}$$

Note that when $\hat{m}_1 = \hat{m}_2$ (or $\tilde{m} = 0$) this result reduces to a familiar $\pi/4$ rotation of field variables. The matrix (23) becomes,

$$U \;\; \rightarrow \;\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{25}$$

which we use in (19) to obtain the simple rotational form of the transformation,

$$\begin{aligned} \Phi_1 &= \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2) \\ \Phi_2 &= \frac{1}{\sqrt{2}}(\varphi_2 - \varphi_1) \end{aligned} \tag{26}$$

To rewrite the Green function in the eigenbasis of the coupled system we solve the equation $\mathcal{O}'\mathcal{G}' = I$. As before, we separate the Green function into the sum of a particular solution and a complementary function: $\mathcal{G}' = G' + 2\pi i \mathcal{F}'$ which satisfy

$$\mathcal{O}' G' = I\,; \quad \mathcal{O}' \mathcal{F}' = 0 \tag{27}$$

The inverse of the diagonal operator $\mathcal{O}'$ is trivial to evaluate. We obtain

$$G' = \begin{pmatrix} G'_{11} & G'_{12} \\ G'_{21} & G'_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{-\Box + m_+^2} & 0 \\ 0 & \frac{1}{-\Box + m_-^2} \end{pmatrix} . \tag{28}$$

In the coupled system, the diagonal form of the complementary function does not have to be preserved. However, we know that:

$$\mathcal{F}' = U \mathcal{F} U^{-1} . \tag{29}$$

In addition, we know that $\mathcal{F}'$ must satisfy (27) which means that it must have the form,

$$\mathcal{F}' = \begin{pmatrix} f'_{11}(k_0)\delta(\chi_+) & f'_{12}(k_0)\delta(\chi_+) \\ f'_{21}(k_0)\delta(\chi_-) & f'_{22}(k_0)\delta(\chi_-) \end{pmatrix} \tag{30}$$

We assume that the original $\hat{\mathcal{F}}$ is diagonal just prior to the interaction begin turned on, so that if the interaction is turned on suddenly, $\mathcal{F}$ remains diagonal:

$$\mathcal{F} = \begin{pmatrix} f_1(k_0)\delta(\chi_1) & 0 \\ 0 & f_2(k_0)\delta(\chi_2) \end{pmatrix} \tag{31}$$

where χ_1 and χ_2 are as yet undetermined mass shells. Using (23) and (31) in (29) and comparing with (30) we obtain the relationship between the components of $\mathcal{F}$ and $\mathcal{F}'$:

$$\begin{aligned} \frac{f'_{11}}{\omega_+} &= \frac{f_1}{\omega_1}\cos^2\theta + \frac{f_2}{\omega_2}\sin^2\theta \\ \frac{f'_{22}}{\omega_-} &= \frac{f_2}{\omega_2}\cos^2\theta + \frac{f_1}{\omega_1}\sin^2\theta \\ \frac{f'_{12}}{\omega_+} &= (\frac{f_2}{\omega_2} - \frac{f_1}{\omega_1})\cos\theta\sin\theta \\ \frac{f'_{21}}{\omega_-} &= (\frac{f_2}{\omega_2} - \frac{f_1}{\omega_1})\cos\theta\sin\theta. \end{aligned} \tag{32}$$

3 Covariant analysis at $\overline{t} \neq 0$

Now we consider the behaviour of the system for $\overline{t} \neq 0$. We make use of the covariant method of [8] to solve for the Wightman or complementary Green functions for the effective theory. The off-diagonal terms are irrelevant to further analysis; they trivially satisfy the equations of motion by virtue of the fact that their particular solutions are identically zero and their complementary parts contain delta-function constraints. For notational simplicity we therefore refer to the diagonal components as $G'_{11} := G'_+$, $G'_{22} := G'_-$, $f'_{11} := f'_+$ and $f'_{22} := f'_-$.

Our strategy is to find the dispersion relation for the positive frequency Wightman function. This dispersion relation does not have a unique form: it depends on the ansatz used for the Green function itself. By using a conformally covariant form for the ansatz and introducing vector fields A_μ, we obtain a form for the dispersion relation which is independent of $\overline{x}$. This approach is useful in two ways: it results in a neater form for the dispersion relation, and it allows the internal conformal symmetry to be seen directly and used to simplify calculations where the dispersion relation is not $\overline{x}$ independent. Note that we are not claiming that the theory is conformally invariant. We are only noting that an overall $\overline{x}$ scaling of the action can be used to express any statistical theory in terms of dimensionless variables, provided partial derivatives are replaced by covariant ones [8]. By formally introducing covariant connections, we can exploit our ability to rescale a slowly varying non-equilibrium theory so that it appears to be an equilibrium one, for the purposes of calculation.

The positive frequency Wightman functions satisfy the equations of motion:

$$(-\overset{x}{\Box} + m_a^2)G_a^{(+)'}(x,x') = 0\,, \tag{33}$$

where the index a takes values $\{+,-\}$. We attempt to find solutions to these equations using the ansatz,

$$G_a^{(+)'}(x,x') = 2\pi i \int (d\mathbf{k})\, e^{ik\tilde{x} + i\int_0^{\overline{x}} A_{a\mu}\, dX^\mu} \frac{(1 + f'_a(\omega_a(\overline{x}),\overline{x}))}{2\omega_a(\overline{x})}\,. \tag{34}$$

The formal dependence of the distribution functions f'_a and on the average coordinate $\overline{x}$ seems peculiar in a free theory, where there is no apparent cause for a re-distribution of quanta amongst different modes in either time or space. Such a change would have to be represented by an additional source or sink (a potential), which we do not have. However, there is nothing in the field equations to exclude a combination of solutions which oscillates with a steady periodic signature. Such a steady-state oscillation could be fed in as a boundary condition. Another example would be the analysis of low-order terms in an interacting theory, where the effective mass was changing with time $m^2(t)$. Such a change makes no sense without a broader theory to explain it, but as an effective model it can still be analyzed in a consistent framework.

The purpose of the auxiliary field A_μ is as follows. Since external boundary conditions and renormalizations might impose a time-dependence in the dispersion relation, one needs a generic term A_μ to absorb the effect of the changing distribution. Its introduction allows us to formulate the theory in terms of a time-independent frequency, by moving the dependence of the average coordinate into the phase.

Taking the derivative we obtain,

$$\begin{aligned} \overset{x}{\partial}_\mu \, G_a^{(+)'}(x,x') &= 2\pi i \int (dk) \Big[ik_\mu + \frac{i}{2} A_{a\mu} + i \int_0^{\overline{x}} (\tilde{\partial}_\mu A_{a\nu}) dX^\nu \\ &- \frac{i}{2} (\overline{\partial}_\mu \omega_a) \tilde{t} + \frac{1}{2} F'_{a\mu} - \frac{1}{2} \Omega_{a\mu} \Big] G_a^{(+)'}(k, \overline{x}) \,, \end{aligned} \tag{35}$$

where,

$$\Omega_{a\mu} = \partial_\mu \omega_a(\bar{x}) / \omega_a(\bar{x}) \,; \quad F'_{a\mu} = \frac{\partial_\mu f'_a}{1 + f'_a} \tag{36}$$

and

$$\begin{aligned} \overset{x}{\partial}_\mu &= \tilde{\partial}_\mu + \frac{1}{2} \overline{\partial}_\mu \\ \overset{x}{\Box} &= \tilde{\Box} + \tilde{\partial}_\mu \overline{\partial}^\mu + \frac{1}{4} \overline{\Box} \,. \end{aligned} \tag{37}$$

Notice that the second term of the first line above behaves as an auxiliary field. It is this term that will be absorbed by the auxiliary field.

We proceed as follows. We make the choice $A_{a\mu} = \tilde{t} \bar{\partial}_\mu \omega_a$. Substituting into $i \int_0^{\bar{x}} (\tilde{\partial}_\mu A_{a\nu}) d\bar{x}'^\nu$ and integrating by parts we obtain,

$$i \int_0^{\bar{x}} (\tilde{\partial}_\mu A_{a\nu}) d\bar{x}'^\nu = i \delta_\mu{}^0 [\omega_a(\bar{x}) - \omega_a(0)]. \tag{38}$$

When we substitute this result back into the dispersion relation we obtain,

$$\partial^x_\mu G_a^{(+)'} = 2\pi i \int (dk) \left[i(k_{a\mu} + \delta_\mu{}^0 [\omega_a(\bar{x}) - \omega_a(0)]) + \frac{1}{2} (F'_{a\mu} - \Omega_{a\mu}) \right] G_a^{(+)'} \tag{39}$$

We re-express this result covariantly by defining the vector $k_{a\mu}(0) = \omega_a(0) \delta_{\mu 0} + k^i \delta_{i\mu}$ which allows us to write

$$\partial_\mu G_a^{(+)'} = 2\pi i \int (dk) \, [i k_{a\mu}(0) + \frac{1}{2} (F'_{a\mu} - \Omega_{a\mu})] G_a^{(+)'} \tag{40}$$

Performing another derivative and substituting into (33) we obtain,

$$- k_{a\mu}(0) k_a^\mu(0) - m_a^2 + \frac{1}{4} [\bar{\partial}^\mu (F'_{a\mu} - \Omega_{a\mu}) + (F'_a - \Omega_a)^2] + i k_a^\mu(0) (F'_{a\mu} - \Omega_{a\mu}) = 0 \tag{41}$$

The first two terms give $-k_a(0)^2 - m_a^2 = \omega_a^2(0) - \mathbf{k}^2 - m_a^2$ which we set to zero as an initial condition on $\omega_a(\bar{x})$. More precisely if ω_a is constant it must equal the equilibrium value for the given mass. Then we are left with two equations

$$\begin{aligned} & k_a^\mu(0)(F'_{a\mu} - \Omega_{a\mu}) = 0 \\ & \bar{\partial}^\mu(F'_{a\mu} - \Omega_{a\mu}) + (F'_a - \Omega_a)^2 = 0 \end{aligned} \tag{42}$$

4 Solutions

In order to look for solutions we set $\Omega_\mu = 0$. The first part of (42) gives

$$k^\mu(0)\partial_\mu f'_a = 0 \tag{43}$$

If we assume spatial homogeneity then this equation reduces to

$$\partial_t f'_a = 0\,. \tag{44}$$

Now we want to ask what this result implies about the distribution functions of the original degrees of freedom. Using (32) we obtain,

$$\partial_t f'_a = 0 \quad \rightarrow \quad \begin{pmatrix} \frac{\omega_+ \cos^2\theta}{\omega_1} & \frac{\omega_+ \sin^2\theta}{\omega_2} \\ \frac{\omega_- \cos^2\theta}{\omega_2} & \frac{\omega_- \sin^2\theta}{\omega_1} \end{pmatrix} \begin{pmatrix} \partial_t f_1 \\ \partial_t f_2 \end{pmatrix} = 0 \tag{45}$$

In general, this implies that the distribution functions in terms of the original degrees of freedom must also be constant. The only way to obtain non-trivial solutions to this equation is to require that the determinant of the matrix of coefficients must vanish. This requires

$$\cos^2\theta\, \sin^2\theta\, \omega_+^2 \omega_-^2 \left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right) = 0\,. \tag{46}$$

Since $\cos\theta$ and $\sin\theta$ can only vanish when the coupling, m_p, vanishes (cf (20)) the only possible solution to this equation is for ω_1 to equal ω_2.

5 A Generalizable Analysis

An explicit rotation of variables is not the approach normally used in solving field theories, since it does not readily generalize to arbitrary higher order interactions. An alternative method, which leads to an effective theory for one of the systems can be found by making a change of variable:

$$\begin{aligned} \Phi_2 &= \phi_2 - \int (dx') G_2(x,x')\, J_2(x') \\ &= \phi_2 - m_p^2 \int (dx') G_2(x,x')\, \Phi_1(x'). \end{aligned} \tag{47}$$

The shift involves a non-local Green function, which is assumed to satisfy

$$(-\Box + \hat{m}_2^2) G_2(x,x') = \delta(x,x'). \tag{48}$$

The boundary conditions on $G_2(x,x')$ are not determined by this equation or by the Schwinger action principle itself. To determine these boundary conditions we use the fact that the interacting fields Φ_1 and Φ_2 are influenced by one another only after they are coupled at time $\overline{t} = 0$. This physical consideration leads us to choose the retarded Green function.

Making the shift of variables (47) in (11) gives,

$$S_{\rm tot} = \int (dx)\Big\{\Phi_1^*(x)(-\Box + \hat{m}_1^2)\Phi_1(x) - m_p^4 \int (dx')\Phi_1^*(x)G_2(x,x')\Phi_1(x') + \phi_2^*(x)(-\Box + \hat{m}_2^2)\phi_2(x)\Big\} \tag{49}$$

We are left with a non-local interaction in terms of the original variables. The resulting equations of motion have the matrix form,

$$\tilde{\mathcal{O}} \begin{pmatrix} \Phi_1 \\ \phi_1 \end{pmatrix} = 0 \tag{50}$$

with

$$\tilde{\mathcal{O}} = \begin{pmatrix} -\Box + \hat{m}_1^2 - \frac{m_p^4}{-\Box + m_2^2} & 0 \\ 0 & -\Box + \hat{m}_2^2 \end{pmatrix} \tag{51}$$

To find the particular part of the corresponding Green function we need to invert the matrix $\tilde{\mathcal{O}}$. We obtain,

$$\tilde{\mathcal{O}}^{-1} = \frac{1}{{\rm det}\tilde{\mathcal{O}}} \begin{pmatrix} -\Box + \hat{m}_2^2 & 0 \\ 0 & -\Box + \hat{m}_1^2 - \frac{m_p^4}{-\Box + m_2^2} \end{pmatrix}. \tag{52}$$

Using

$${\rm det}\tilde{\mathcal{O}} = (-\Box + \hat{m}_1^2)(-\Box + \hat{m}_2^2) - m_p^4 = (-\Box + m_+^2)(-\Box + m_-^2) \tag{53}$$

it is clear that the Green function has poles at the excitation masses m_+ and m_-. This result agrees with the result found in the previous section: the true mass shell of the excitations is given by the eigenmasses of the system.

In order to find the complementary part of the Green function we follow the method of the previous section. We define the matrix

$$\Lambda = \begin{pmatrix} 1 & 0 \\ \frac{-m_p^2}{-\Box + \hat{m}_2^2} & 1 \end{pmatrix} \tag{54}$$

The equation that is analogous to (19) is then

$$\Phi = \Lambda\phi\,. \tag{55}$$

We can obtain the diagonalized matrix $\tilde{\mathcal{O}}$ given in (51) by performing the transformation on $\mathcal{O}$ analogous to (16):

$$\tilde{\mathcal{O}} = (\Lambda^\dagger)^{-1}\mathcal{O}\Lambda^{-1}\,. \tag{56}$$

Using $\mathcal{O}$ as defined in (14) we reproduce (51). The complementary piece of the Green function is obtained from the equation analogous to (16):

$$\tilde{\mathcal{F}} = (\Lambda^\dagger)^{-1}\mathcal{F}\Lambda^{-1}\,. \tag{57}$$

To proceed we rewrite (51) in the form

$$\tilde{\mathcal{O}} = \begin{pmatrix} \frac{1}{-\Box + \hat{m}_2^2}(-\Box + m_+^2)(-\Box + m_-^2) & 0 \\ 0 & (-\Box + m_2^2) \end{pmatrix}, \tag{58}$$

and impose $\tilde{\mathcal{O}}\tilde{\mathcal{F}} = 0$ by writing

$$\tilde{\mathcal{F}} = \begin{pmatrix} \tilde{f}_{11}\delta(\chi_\pm) & \tilde{f}_{12}\delta(\chi_\pm) \\ \tilde{f}_{21}\delta(\hat{\chi}_2) & \tilde{f}_{22}\delta(\hat{\chi}_2) \end{pmatrix} \tag{59}$$

where $\delta(\chi_\pm)$ indicates any linear combination of $\delta(\chi_+)$ and $\delta(\chi_-)$. Using (31) and (54) in (57) and comparing with (59) we obtain,

$$\begin{aligned} \tilde{f}_{22}\delta(\hat{\chi}_2) \quad &= f_2\delta(\chi_2) \\ \tilde{f}_{11}\delta(\chi_\pm) \quad &= f_1\delta(\chi_1) + \frac{m_p^4}{(-\overset{x}{\Box} + \hat{m}_2^2)^2} f_2\delta(\chi_2) \end{aligned} \tag{60}$$

In order to study the time dependence of the distribution functions $\tilde{\mathcal{F}}$ we proceed as in the previous section. The $\phi_2(x)$ Wightman function obeys the equation of motion,

$$(-\overset{x}{\Box} + \hat{m}_2^2)\tilde{G}_2^{(+)}(x, x') = 0, \tag{61}$$

and its dispersion constraint is the free mass-shell

$$\hat{\chi}_2 = k^2 + \hat{m}_2^2 = 0. \tag{62}$$

We postulate an ansatz of the form,

$$\tilde{G}_2^{(+)}(x, x') = 2\pi i \int (d\mathbf{k}) e^{ik\tilde{x} + i\int_0^{\overline{x}} A_{2\mu}\, dX^\mu} \frac{(1 + \tilde{f}_{22}(\omega_{22}(\overline{x}), \overline{x}))}{2\omega_{22}(\overline{x})}. \tag{63}$$

Substituting into (61) gives,

$$\partial_t \tilde{f}_{22} = 0 \tag{64}$$

Using (60) we get $\partial_t f_2 = 0$ or $f_2 =$ constant.

The $\Phi_1(x)$ Wightman function is more complex; it satisfies the equation of motion,

$$(-\overset{x}{\Box} + \hat{m}_1^2)\tilde{G}_1^{(+)}(x, x') - m_p^4 \int (dx'') G_2(x, x'')\tilde{G}_1^{(+)}(x'', x') = 0\,. \tag{65}$$

To simplify this expression we operate on the left with $(-\overset{x}{\Box} + \hat{m}_2^2)$ which gives,

$$[(-\overset{x}{\Box} + \hat{m}_2^2)(-\overset{x}{\Box} + \hat{m}_1^2) - m_p^2]\tilde{G}_1^{(+)} = (-\overset{x}{\Box} + m_+^2)(-\overset{x}{\Box} + m_-^2)\tilde{G}_1^{(+)} = 0 \tag{66}$$

We postulate an expression for $\tilde{G}_1^{(+)}(x, x')$ of the form,

$$\tilde{G}_1^{(+)}(x, x') = 2\pi i \int (d\mathbf{k}) e^{ik\tilde{x} + i\int_0^{\overline{x}} A_{1\mu}\, dX^\mu} \frac{(1 + \tilde{f}_{11}(\omega_{11}(\overline{x}), \overline{x}))}{2\omega_{11}(\overline{x})}. \tag{67}$$

Substituting (67) into (66) we could solve for the dispersion relation, as in the previous section. The formalism discussed in this section could be used to study a system with a more complicated non-linear interaction, which would lead to non-trivial solutions.

6 Conclusion

We have studied a system which consists of two initially independent scalar systems that are minimally coupled at some initial time. We have used the machinery of Green functions and effective field theory to study the time evolution of the statistical distribution functions. This problem is an inherently non-equilibrium one. We have discussed how to parameterize such an equilibrating system, and how to perform calculations when equations of motion and thus dispersion relations change with time. Building on the earlier work in [5, 8] we have adopted a covariant approach to handle time dependent dispersion relations and used the underlying conformal structure to show the consistency of working with perturbation theory in which dispersion relations are time-independent. We have shown that by formally introducing covariant connections, one can rescale a slowly varying non-equilibrium theory so that it appears to be an equilibrium one, for the purposes of calculation. This approach turns out to be of only formal utility in the example provided, but can be generalized to more complex and realistic models. The simple minimally coupled system studied here does not reach a conventional kind of macroscopic equilibrium. In order to study more conventional equilibration, in which the statistical properties are related to the mechanical interactions, we would need to include higher order interactions.

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